MEC 3458 EXPERIMENTAL PROJECTS

ERROR & ERROR ANALYSIS

TOPIC 1: INTRODUCTION & DEFINITIONS

D. HONNERY 2008.

WHAT IS ERROR?

ERROR IS THE DIFFERENCE BETWEEN A MEASURED VALUE & THE TRUE VALUE OF THE MEASUREMENT.

FOR EXAMPLE WE MAY WRITE,

$$\text{ERROR} = E = X_T - X_M$$

WHERE X_T IS THE TRUE VALUE & X_M THE MEASURED. HERE E IS THE ABSOLUTE ERROR

WE COULD WRITE THE ERROR AS A RELATIVE ERROR,

$$e = \frac{E}{X_m} = \frac{X_T - X_M}{X_M} = \frac{X_T}{X_m} - 1$$

OR WE COULD WRITE THE ERROR AS AN ERROR TOLERANCE. HERE TOLERANCE REFERS TO THE RANGE BETWEEN THE

MAXIMUM & MINIMUM ERROR IN WHICH THE TRUE VALUE CAN BE FOUND,

FOR δ_{MIN} = MINIMUM ERROR LIMIT

δ_{MAX} = MAXIMUM ERROR LIMIT

THEN

$$(X_m - \delta_{min}) \leq X_T \leq (X_m + \delta_{MAX})$$

THUS $X_T = X_{m-\delta_{min}}^{+\delta_{MAX}}$

FOR $\delta_{max} = \delta_{MIN} = \delta$ THE ERROR IS SYMMETRICAL

THUS $X_T = X_m \pm \delta$

FOR EXAMPLE WE COULD WRITE A TEMPERATURE AS

$$T = 50 \pm 1{\cdot}5\ ^{\circ}C$$

OR

$$T = 50^{\circ}C \pm 3{\cdot}0\%$$

WHAT ARE ACCURACY & PRECISION?

IF ERROR IS THE DIFFERENCE BETWEEN THE MEASURED & TRUE VALUE, ACCURACY IS A MEASURE OF HOW CLOSE THE MEASURED VALUE IS TO THE TRUE VALUE. ACCURACY IS SIMILAR TO A BIAS IN A MEASUREMENT.

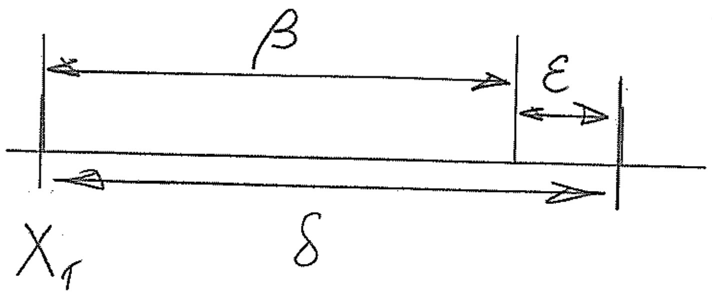
PRECISION REFERS TO THE SIZE OF THE CONTRIBUTION OF RANDOM INFLUENCES ON THE MEASUREMENT.

CONSIDER THE FIGURE SHOWN OVER

THE TOTAL ERROR δ IS MADE UP OF BOTH BIAS & PRECISION ERRORS

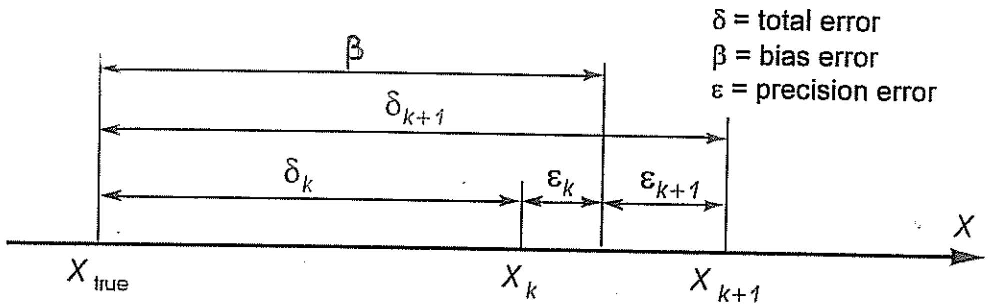
$$\left.\begin{array}{l}\beta = \text{BIAS ERROR} \\ \varepsilon = \text{PRECISION ERROR}\end{array}\right\} \Longrightarrow \delta \text{ TOTAL ERROR}$$

FOR TRUE VALUE X_T WITH BIAS & PRECISION ERRORS β & ε WE MAY WRITE



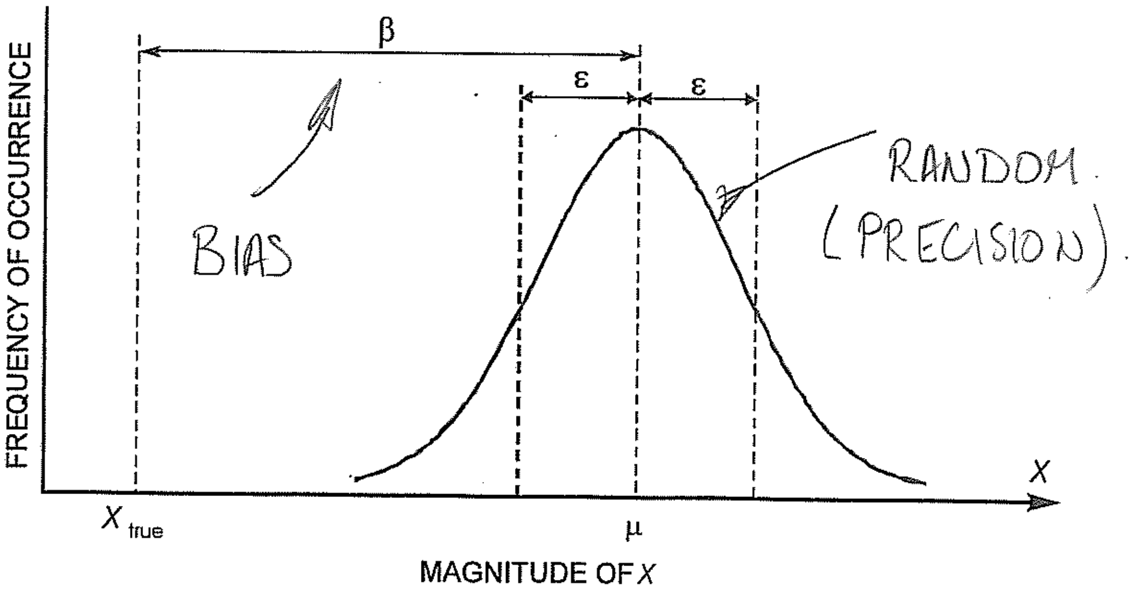
PRECISION ERRORS OFTEN CALLED A-TYPE; BIAS ERRORS B-TYPE

FOR TWO MEASUREMENTS OF X_T (X_{TRUE})



(a) two readings

FOR MANY MEASUREMENTS,



(b) infinite number of readings

THUS BIAS ERRORS ARE FIXED. PRECISION ARE NOT.

EXAMPLES OF BIAS ERRORS ARE THE RESOLUTION OF THE INSTRUMENT, EG A RULER READS IN 'mm', THERMOMETER 0·1°C.

FOR N SAMPLES OF A MEASUREMENT, THE DIFFERENCE BETWEEN THE AVERAGE VALUE & THE TRUE VALUE IS THE BIAS ERROR.

BIAS ERRORS CANNOT BE DETERMINED STATISTICALLY.

SOME MANUFACTURERS PROVIDE INFORMATION ON BIAS ERROR WITH EQUIPMENT, EG

PRESSURE TRANSDUCER READS $\pm$ 0·1 Pa.

SAMPLES OF TYPICAL BIAS ERRORS ARE GIVEN IN THE FOLLOWING TABLES.

TYPICAL BIAS ERRORS FOR A RANGE OF INSTRUMENTS.

FROM GIBBINGS, THE SYSTEMATIC EXPERIMENT
CAMBRIDGE PRESS 1986.

Table 5.2. *Summary of commonly used laboratory equipment*

Parameter to be measured	Measuring device	Typical range	Approximate resolution	Remarks
Length	Machinist's calipers with steel rule	0–50 mm up to 0–1.0 m	±0.2 mm	Used mainly for checking nominal dimensions
	Vernier calipers or height gauge	0–0.1 m up to 0–0.5 m	±0.02 mm	
	Micrometer	0–25 mm or 25–50 mm, etc.	±0.005 mm	Range can be extended considerably to, say, 1 m ± 12.5 mm by use of a standard 25 mm micrometer head and a 'bow gauge' structure
	Slip gauges	e.g. 0.5–100.0 mm	±0.001 mm	Used mainly for the calibration of lower precision instruments
Displacement	Dial gauge	e.g. 0–5 mm 0–50 mm	±0.01 mm	Extensively used, particularly in workshop practice
	Linear variable differential transformer	e.g. 0–0.25 mm up to 0–0.3 m	±0.1%	Resolution theoretically infinite but in practice dependent on associated instrumentation
Acceleration	Accelerometer	e.g 0–7000 g	0.05 g	With suitable instrumentation can be used for recording periodic displacements
Strain	Mechanical extensometers	e.g. Huggenberger type 0–0.4%	$\pm 10 \times 10^{-6}$	Range can be increased by resetting
	Electrical resistance strain gauges	±1.5% up to ±20% strain	$\pm 1 \times 10^{-6}$	Generally useful in transducer applications for measurement of displacement, load, torque, pressure, etc. Not re-usable and therefore costly

Table 5.2 (*cont.*)

Parameter to be measured	Measuring device	Typical range	Approximate resolution	Remarks
Time and frequency	Stopclocks	e.g. 0–60 min	± 0.2 s	Not suitable for short periods of time, say, < 10 s. Stopwatches available with better resolution
	Tachometer (mechanical or electrical)	e.g. 0–50 000 rpm	–	
	Stroboscope	e.g 0–100 Hz 0–300 Hz	–	Especially useful for shaft speed or frequency measurements, where it is undesirable to have direct contact with the moving part
	Cathode ray oscilloscope (cro) with transducer	e.g. 0.5 μs–1 s dc to 3 MHz	$\pm 5\%$ –	By far the most widely used item of equipment for examination of transient and repetitive signals
	Electronic frequency counters with transducer	e.g. 1 μs–10^4 s 10 Hz–1.2 MHz	e.g. ± 1 μs –	Very versatile instrument for use with any form of wave or pulse pickup. Can be used for time or frequency measurement
	Trace recorders with transducer	0–2 Hz up to 0–5 kHz	$\pm 1\%$	Pen recorders are used for low frequency response; ultraviolet recorders, with suitable galvanometers, for high frequency response. Adjustable chart speed
Mass	Balances	e.g 0–20 g → 0–200 g → 0–5 kg →	± 0.001 mg ± 1 mg ± 4 mg	For extreme accuracy 'ultra-microbalances' are available with a resolution of ± 0.1 μg
Force	Spring balance	e.g. 0–20 N up to 0–5 kN	$\pm 1.0\%$	Resolution often poorer in the simpler spring balance arrangements
	Proving rings	e.g. 0–0.2 kN up to 0–70 kN	$\pm 0.1\%$	Accuracy dependent on technique used for measuring ring deflection

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BIAS ERRORS CON'T

	Strain gauge load cell	e.g. 0–40 N up to 0–1 MN	±0.1%	If not available commercially, strain gauge load cells can usually be devised to provide the required force measurement (see Table 5.1)
	Hydraulic or pneumatic ram	Any range up to about 2 MN	See comments	Accuracy very much dependent on friction effects, but also on pressure-measuring system
	Piezoelectric force transducer	e.g. 0–±8 kN up to 0–±5 MN	±20 mN	These transducers noted for their high resolution
Torque	Prony brakes	General principle suitable for most laboratory power units	See comments	Used for power measurement in rotating shafts. Accuracy dependent on force measuring technique, etc.
	Torsion dynamometer	0–10 mNm up to 0–5 kNm	±0.1%	This type suitable for either 'static' or 'rotating' torque measurement
Pressure	U-tube manometer	20 mm–5 m of manometer fluid (vertical U-tube)	±0.5 mm ±0.05 mm at 5° to horizontal	Range can be increased but it is not usually practicable or convenient. Lower limit of range can be reduced to about 2 mm by tilting tubes to 5° from horizontal
	Micromanometer	0–0.2 m of manometer fluid	±0.002 mm	A high-precision instrument
	Bourdon tube	0–0.1 MN m^{-2} up to 0–500 MN m^{-2}	±1%	Combined negative and positive ranges sometimes used on low pressure gauges
	Dead-weight tester	100 N m^{-2} 100 MN m^{-2}	±0.01 to ±0.05%	Precision depends upon quality of device. Used for calibrating other gauges
	Pressure transducers	0.002 mm of water to 500 MN m^{-2}	±0.1%	Sensitivity to low pressure differences limited by friction or by thermal expansion
Flow velocity	Pitot–static tube	6–60 m s^{-1} usually	±1% of dynamic pressure	Quoted accuracy applies to International Standard designs. Possible to extend down to 1 or 2 m s^{-1}. Can be used up to Mach 5

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Table 5.2 (*cont.*)

Parameter to be measured	Measuring device	Typical range	Approximate resolution	Remarks
	Hot-wire anemometer	Up to 250 m s^{-1} in filtered air at temperatures up to 150 °C. Up to 5 m s^{-1} in liquids	1 mm	Less accurate than pitot–static method except at very low velocities. Has very fast response – is very good for unsteady flow and turbulence measurements. Fragile
	Hot-film anemometer	50 m s^{-1} in gases 25 m s^{-1} in liquids	1 mm	Compared with hot wires they are much more robust, less susceptible to contamination but have a poorer frequency response
	Laser anemometer	—	0.2 mm cube	No physical interference in fluid but requires 'particles' 2–5 μm to give good Doppler signals
Flow direction	Wedge yawmeter	—	$\pm 0.2°$ or less	Also measures velocity. Works well in transverse gradients of total pressure. Small size
Mass or volume flow rate	Orifice plate	See remarks	$\pm 1\%$	Simple, cheap but has large losses in pressure. Size of plate or tube chosen to give a differential pressure of 0.5–5 m of fluid on a U-tube manometer
	Venturi meter	See remarks	$\pm 1\%$	Fairly expensive but has low pressure losses
	Vortex shedding flowmeter	1–30 m s^{-1} in liquids	$\pm 1\%$	Good repeatability, low cost, linearity, low pressure drop, rapid response

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BIAS ERRORS CON'T

Temperature	Mercury in glass thermometer	−35–510 °C	$\pm\frac{1}{2}$ °C	High temperature can cause ageing in glass. Zero calibration needed (in melting ice)
	Alcohol in glass thermometer	−80–100 °C		
	Pentane in glass thermometer	−200–30 °C		
	Mercury in steel thermometer with Bourdon gauge	−35–650 °C	±1%	Useful for remote readings. Correction for differences in level required
	Bimetal thermometer	−180–450 °C	±1%	Slow response
	Platinum resistance thermometer	−240–1060 °C	±0.2 °C ($0 \leqslant T \leqslant 100$ °C) ±0.35 °C (at 200 °C) ±0.55 °C (at 300 °C) ±0.8 °C (at 400 °C)	Very high precision – often used for interpolating between primary standard temperature points. See BS 1904 for temperature-resistance relationship and tolerances
	Thermocouples			
	Chromel and constantan	−180–1000 °C		
	Copper and constantan	−180–400 °C	±5 °C	Simple and cheap
	Iron and constantan	−180–850 °C	±10 °C	Use in non-oxidising atmosphere
	Chromel and alumel	0–1100 °C	±10 °C	
	Tungsten and rhenium	0–2800 °C		
	Platinum and rhodium	0–1450 °C	±3 °C	Not subject to corrosion – very reliable – expensive
	Optical pyrometers (disappearing filament type)	750–4000+ °C	±4 °C (at 1000 °C) ±6 °C (at 2000 °C) ±40 °C (at 4000 °C)	When calibrated against a standard. Glass absorption filter used above about 1350 °C
Voltage and current	Moving coil and moving iron instruments, etc.	See comments	±0.5% for precision grade ±1–±3% for industrial grade	Lower limit of effective range varies according to type of movement (typically this is 10% to 30% of full scale). See BS 89 for further information

General note: Any one instrument may cover only part of the ranges shown in the table. Ranges and resolution quoted are for guidance and may differ between manufacturers of the same type of instrument.

PRECISION ERRORS VARY FOR EACH MEASUREMENT. REPEAT MEASUREMENT WITH THE SAME INSTRUMENT WILL YIELD SCATTER IN THE MEASUREMENT, THE SCATTER IS DUE TO PRECISION ERROR.

PRECISION ERROR <u>CAN BE</u> DETERMINED STATISTICALLY.

PRECISION ERROR IS PROPORTIONAL TO THE STANDARD DEVIATION OF A SET OF MEASUREMENTS.

<u>WHAT IS UNCERTAINTY?</u>

UNCERTAINTY REFERS TO THE ESTIMATION OF ERROR.

DETERMINING THE ERROR IN A MEASUREMENT REQUIRES KNOWING THE TRUE VALUE. THE TRUE VALUE IS RARELY KNOWN, THUS WE ESTIMATE ERROR. THIS ESTIMATION IS THE UNCERTAINTY. IT IS USUAL TO EXPRESS THIS

UNCERTAINTY WITHIN LIMITS OF CONFIDENCE.

CONFIDENCE IS RELATED TO PROBABILITY.

THE TRUE VALUE FALLS WITHIN THE RANGE $\pm S$ AT THE 95% CONFIDENCE LIMIT. THAT IS 19 TIMES OUT OF 20 MEASUREMENTS (95%) THE TRUE VALUE WILL BE IN THE RANGE $\pm S$.

WHAT ARE SYSTEMATIC ERRORS?

SYSTEMATIC ERRORS ARE SIMILAR TO BIAS ERRORS BUT WE DIVIDE THEM INTO TWO CLASSES. THOSE THAT RELATE TO THE RESOLUTION OF THE INSTRUMENT & THOSE THAT DON'T. WE CONSIDER ONLY THOSE THAT RELATE TO RESOLUTION, THOSE LISTED BELOW WE IGNORE.

SYSTEMATIC ERRORS RESULTING FROM NON-RESOLUTION INSTRUMENT SOURCES.

(1) *Method errors:* these errors arise when a wrong or insufficient experimental method has been chosen. Measuring of one quantity in mistake for another may occur or some unrecognised effects may influence the quantity measured so that the resultant values become erroneous. Unjustified extrapolation of experimental data may also lead to method errors.

(2) *Instrument errors:* errors of this type can be caused by a faulty instrument, the misoperation of an instrument, or by using an instrument in the environment for which it was not designed. The instrument errors are frequently biased in one direction, although in some situations hysteresis effects can occur (e.g. a worn-out micrometer, or traverse mechanism).

(3) *Calibration errors:* most instruments will not yield correct results unless they are calibrated before use against a known quantity. This may involve a simple zero setting or determination of a whole calibration curve (or scale). In either case errors can creep into the calibration procedure.

(4) *Human errors:* human errors depend on the personal characteristics of the observer. A human may respond to a signal too early or too late; he may either overestimate or underestimate the reading. Such errors are usually fairly consistent as they are committed perpetually by the same observer at a single session. Occasional errors committed spasmodically, owing to relaxation of vigilance for example, do not apply here and are classed as mistakes.

(5) *Arithmetic errors:* arithmetical calculations involved in experimentation are nowadays being increasingly taken over by various automatic computing devices (computers, automatic desk calculators, slide rules, etc.). Aberrations of such devices, however infrequent, cannot be ruled out completely. Additionally, there may be faults in the actual calculation procedures (programs). Incorrect rounding off can also contribute to arithmetical errors.

(6) *Dynamic response errors:* it is perhaps slightly out of place to devote a separate section to the dynamic response errors. However, their significant participation in modern experimentation, especially in connection with measuring time-dependent variables, warrants a separate emphasis. Unlike static response errors (static non-uniformity of action, hysteresis), dynamic response errors arise when an instrument recording a fast changing signal fails to respond linearly to the signal variation (e.g. a pitot–static tube in fluctuating fluid flows, or electrical instruments applied to non-sinusoidal electrical currents, etc.).

FROM GIBBINGS

MEC3458 EXPERIMENTAL PROJECTS

ERROR & ERROR ANALYSIS

TOPIC 2: INTRODUCTION TO STATISTICS.

D-HONNERY 2008

BASIC STATISTICS

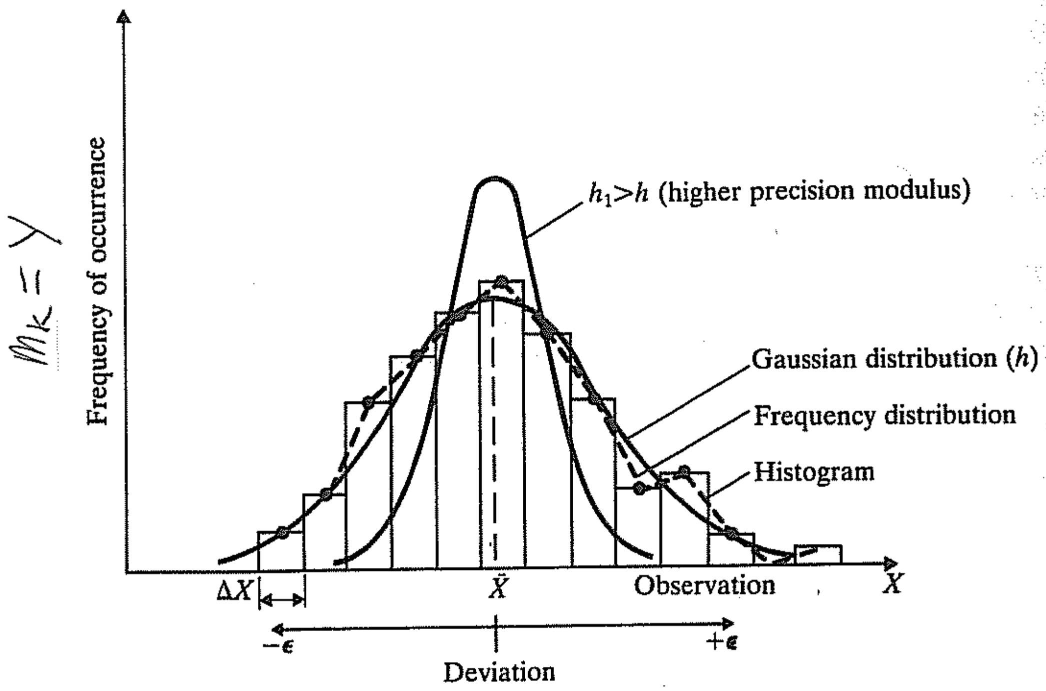
PRECISION ERROR CAN BE ESTIMATED FROM STATISTICAL ANALYSIS. HERE WE REVIEW A NUMBER OF BASIC STATISTICAL DEFINITIONS & RELATIONS.

– FREQUENCY & HISTOGRAMS

FOR A SUFFICIENTLY LARGE SAMPLE OF MEASUREMENTS WE CAN REPRESENT THE SCATTER IN THE DATA AS A HISTOGRAM OR FREQUENCY DISTRIBUTION.

FOR EXAMPLE FOR $X_1, X_2, X_3, \ldots\ldots X_k, \ldots\ldots, X_N$ MEASUREMENTS WE CAN DEFINE A FREQUENCY DISTRIBUTION BY DEFINING AN INTERVAL ΔX_k CENTRED ON X_k UP TO N NUMBER OF INTERVALS: BY COUNTING THE NUMBER OF MEASUREMENTS THAT FALL INTO THE INTERVAL $X_k - \frac{\Delta X_k}{2} \leq X_k < X_k + \frac{\Delta X_k}{2}$ A FREQUENCY OF OCCURENCE m_k FOR EACH ΔX_k CAN BE FOUND.

GRAPHING m_k FOR EACH ΔX_k FOR N INTERVALS YIELDS A FREQUENCY DISTRIBUTION



AS PRESENTED HERE THE FREQUENCY DISTRIBUTION IS THE SOLID LINE JOINING THE INDIVIDUAL M_k VALUES. THE HISTOGRAM IS THE BAR GRAPH OF THE M_k VALUES.

FROM THIS DISTRIBUTION WE MAY NOW DEFINE A NUMBER OF BASIC PROPERTIES.

MEAN OR SAMPLE AVERAGE

$$\bar{X} = \frac{X_1 + X_2 + \dots X_n}{n} = \frac{1}{n}\sum_{k=1}^{n} X_k$$

THIS IS DIFFERENT FROM WHAT IS KNOWN AS THE POPULATION OR ABSOLUTE OR TRUE MEAN.

MEASUREMENT IS LIMITED TO A SAMPLE SIZE n & THE MEAN OF THE SAMPLE IS LIMITED BY THE SAMPLE SIZE IF n IS SMALL. AS $n \to \infty$ THE SAMPLE MEAN APPROACHES THE POPULATION MEAN, μ. FOR A LARGE SAMPLE $\bar{X} \approx \mu$.

THE <u>DEVIATION</u> OF A MEASUREMENT X_k FROM THE MEAN $\bar{X}$ OF THE SAMPLE IS GIVEN BY,

$$\varepsilon_k = X_k - \bar{X}.$$

BY DEFINITION THE MEAN OF THE SAMPLE DEVIATIONS $\bar{\varepsilon} = 0$: THIS IS NOT HELPFUL!.

THE DEVIATION IS EXPRESSED AS EITHER

A ROOT MEAN SQUARE (RMS) OF THE SAMPLE

$$D = \left(\frac{\varepsilon_1^2 + \varepsilon_2^2 + \ldots + \varepsilon_n^2}{n} \right)^{1/2} = \left[\frac{1}{n} \sum_{k=1}^{n} \varepsilon_k^2 \right]^{1/2}$$

OR AS THE <u>SAMPLE STANDARD DEVIATION</u>

$$S = \left[\frac{1}{(n-1)} \sum_{k=1}^{n} \varepsilon_k^2 \right]^{1/2} = \left(\frac{n}{n-1} \right)^{1/2} D.$$

USE OF $(n-1)$ RATHER THAN (n) IS MADE BECAUSE THE SAMPLE MEAN $\bar{X}$ SHOULD BE REMOVED FROM THE CALCULATION OF DEVIATION. THUS IF n IS THE SAMPLE NUMBER, THE STANDARD DEVIATION HAS $(n-1)$ DEGREES OF FREEDOM & THE SAMPLE MEAN HAS (n) DEGREES OF FREEDOM.

JUST AS WITH A MEAN, THERE IS A

SAMPLE STANDARD DEVIATION S FROM A SAMPLE MEAN $\bar{X}$, THERE IS A POPULATION STANDARD DEVIATION σ FROM THE POPULATION MEAN μ. AS $n \to \infty$, $S \to \sigma$. ($\&$ $S \to D$)

THE <u>VARIANCE</u> OF A SAMPLE IS DEFINED AS THE SQUARE OF THE STANDARD DEVIATION,

$$v = S^2.$$

<u>GAUSSIAN DISTRIBUTION</u>

THE FREQUENCY DISTRIBUTION OF THE MEASUREMENT SAMPLE ALMOST ALLWAYS TAKES THE FORM OF A GAUSSIAN DISTRIBUTION (AS MOST UN BIASED SAMPLES DO).

FOR Y REPRESENTING THE FREQUENCY OF THE SAMPLE IN AN INTERVAL NOW DEFINED BY THE DEVIATION ε SUCH THAT $\varepsilon = X - \mu$

THE GAUSSIAN DISTRIBUTION TAKES THE FORM

$$Y(\varepsilon) = \frac{h}{\sqrt{\pi}} e^{-h^2 \varepsilon^2}$$

WHERE h (SEE FIGURE ABOVE) REPRESENTS THE SPREAD OF THE DISTRIBUTION.

WITHIN THE GAUSSIAN DISTRIBUTION WE NOW DEFINE THE SAMPLE INTERVAL AS $\Delta\varepsilon$ SUCH THAT $\varepsilon \pm \Delta\varepsilon/2$. THE NUMBER OF SAMPLES m IN THE INTERVAL $\varepsilon \pm \Delta\varepsilon/2$ GIVES THE PROBABILITY OF THE SAMPLE FALLING IN THE INTERVAL

$$P(\varepsilon) = m/n$$

WHERE n IS THE SAMPLE NUMBER

THUS $P(\varepsilon)$ IS THE PROBABILITY THAT x WILL DEVIATE FROM $\bar{x}$ BY $\varepsilon \pm \Delta\varepsilon/2$

THUS THE GAUSSIAN DISTRIBUTION TAKES THE FORM OF A PROBABILITY DISTRIBUTION.

INTEGRATION OF THE FUNCTION TO FIND THE POPULATION STANDARD DEVIATION YIELDS

$$\sigma = \frac{1}{h\sqrt{2}}$$

THUS $h = 1/\sigma\sqrt{2}$

IT IS USUAL TO WRITE THE FUNCTION BY WRITING

$$Z = \frac{X - \mu}{\sigma} \qquad \mu = \text{POPULATION MEAN}$$

THUS

$$Y(Z) = \sigma Y(\varepsilon) = \frac{1}{\sqrt{2\pi}} e^{-Z^2/2}$$

IF LIMITS OF $-\infty$ TO $+\infty$ ARE SET FOR Z WE MAY SAY THAT

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-z^2/2} \, dz = 1$$

THUS THE PROBABILITY OF FINDING z WITHIN THE POPULATION IS 100%.

LIMITS CAN BE INTRODUCED. CONSIDER FINDING THE DEVIATION ε WITHIN THE LIMITS a_1 TO a_2. THIS CAN BE WRITTEN AS FINDING z WITHIN α_1 TO α_2, THUS,

$$A = \frac{1}{\sqrt{2\pi}} \int_{\alpha_1}^{\alpha_2} e^{-z^2/2} \, dz \qquad \begin{cases} \alpha_1 = a_1/\sigma \\ \alpha_2 = a_2/\sigma \end{cases}$$

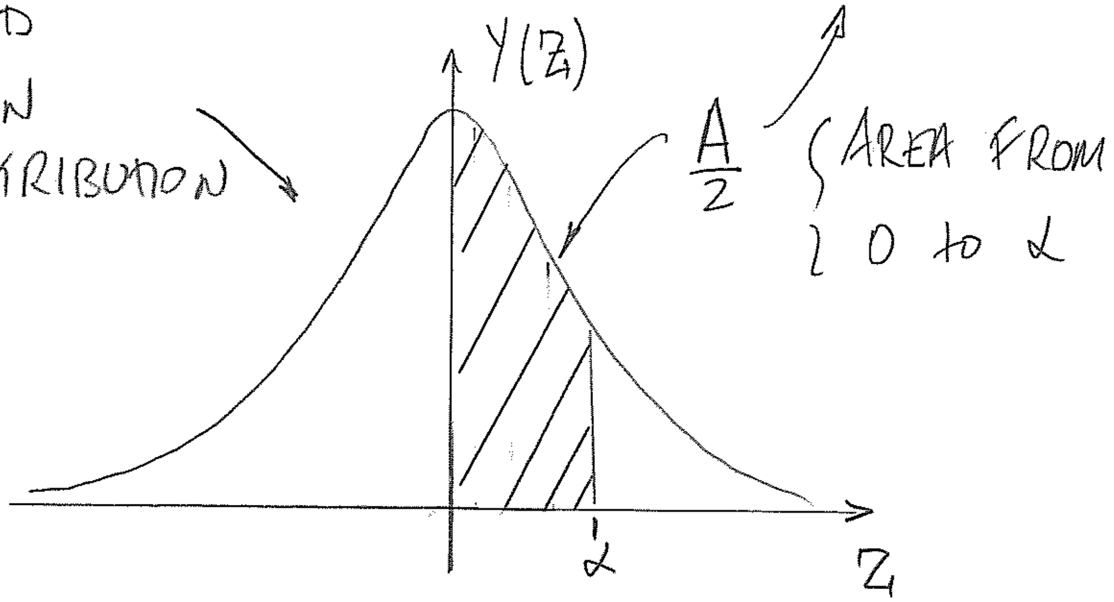
WHERE A IS THE AREA WHICH IS EQUAL TO THE PROPABILITY OF FINDING ε WITHIN LIMITS a_1 TO a_2.

INTEGRATION OF THIS FUNCTION HAS NO SOLUTION OTHER THAN NUMERICAL & IT IS NORMALLY TABULATED, SEE BELOW

$$\frac{A}{2} = \frac{1}{\sqrt{2\pi}} \int_0^{\alpha} e^{-z^2/2}\, dz$$

α	$\frac{1}{2}A$	α	$\frac{1}{2}A$	α	$\frac{1}{2}A$	α	$\frac{1}{2}A$
0	0.0000	0.60	0.2257	1.70	0.4554	2.90	0.49813
0.05	0.01994	0.70	0.2580	1.80	0.4641	3.00	0.49865
0.10	0.03983	0.80	0.2881	1.90	0.4713	3.10	0.49903
0.15	0.05962	0.90	0.3159	2.00	0.4772	3.20	0.49931
0.20	0.07926	1.00	0.3413	2.10	0.4821	3.30	0.49952
0.25	0.09871	1.10	0.3643	2.20	0.4861	3.40	0.49966
0.30	0.1179	1.20	0.3849	2.30	0.4893	3.50	0.49977
0.35	0.1368	1.30	0.4032	2.40	0.49180	3.60	0.49984
0.40	0.1554	1.40	0.4192	2.50	0.49379	3.70	0.49989
0.45	0.1736	1.50	0.4332	2.60	0.49534	3.80	0.49993
0.50	0.1915	1.60	0.4452	2.70	0.49653	3.90	0.49995
				2.80	0.49744	4.00	0.49997

NORMALISED GAUSSIAN DISTRIBUTION



THE TABULATED DATA CAN BE INTERPRETED TO MEAN THE FOLLOWING:

FOR LIMITS $\alpha = \pm 1$ DEVIATIONS FROM THE MEAN RANGE FROM $-\sigma \leq \varepsilon \leq \sigma$.

FROM THE TABE FOR $\alpha = 1$ $\frac{1}{2}A = 0.3413$

THUS $\alpha = \pm 1$ $A = 0.6826$

THUS 68.26% OF THE POPULATION WILL FALL WITHIN THE RANGE OF $\pm\sigma$ OF THE MEAN, μ,

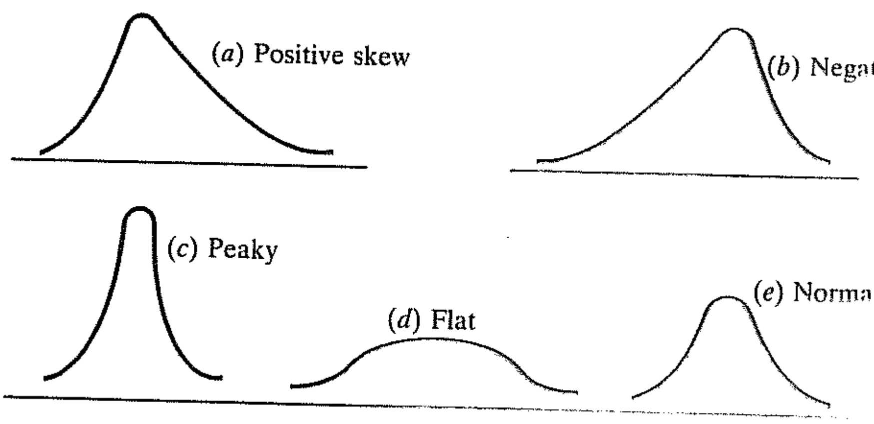
95% OF THE POPULATION REQUIRES $\pm 2\sigma$*

99% OF THE POPULATION REQUIRES $\pm 3\sigma$

* FOR 95%, 5% OF THE POPULATION LIES OUTSIDE $\pm 2\sigma$ FROM μ, ETC.

VARIATION IN GAUSSIAN DISTRIBUTIONS

THE h- FACTOR HAS ALREADY BEEN INTRODUCED WHERE $h = 1/\sigma\sqrt{2}$. THE SMALLER σ THE HIGHER h & THUS THE DISTRIBUTION BECOMES MORE 'PEAKY'. CLEARLY THE RANGE IN THE POPULATION DEVIATION REDUCES. FOR A MEASUREMENT SAMPLE. (WE WOULD SAY THE PRECISION INCREASES, THUS h INDICATES PRECISION.)



VARIOUS FORMS OF DISTRIBUTIONS.

DISTRIBUTIONS (a) & (b) ARE SAID TO BE SKEWED. AND THEREFORE NON-SYMMETRICAL AROUND THE

MEAN. THE DEGREE OF SKEWNESS IS DEFINED BY A SKEWNESS FACTOR λ,

$$\lambda = \frac{1}{nD^3} \sum_{k=1}^{n} \varepsilon_k^3 \qquad \begin{cases} D = \left[\frac{1}{n} \sum_{k=1}^{n} \varepsilon_k^2 \right]^{1/2} \\ \varepsilon_k = X_k - \bar{X} \end{cases}$$

THE FLATNESS OF A DISTRIBUTION IS GIVEN BY THE KURTOSIS (FLATNESS FACTOR) K

$$K = \frac{1}{nD^4} \sum_{k=1} \varepsilon_k^4 .$$

A DISTRIBUTION IS CONSIDERED NORMAL FOR

$$-0.3 < \lambda < 0.3$$

$$2.5 < K < 3.5 .$$

STANDARD ERROR

IF RANDOM SAMPLES OF SIZE n ARE DRAWN FROM A POPULATION WITH MEAN μ & STANDARD DEVIATION σ, THEN THE MEAN OF THE SAMPLE MEANS IS $(\overline{X})_i = \mu$. THE STANDARD DEVIATION OF THE SAMPLE MEANS IS KNOWN AS THE STANDARD ERROR σ_m

$$\sigma_m = \frac{\sigma}{\sqrt{n}} \qquad (\text{FOR } n > 20).$$

SIGNIFICANCE TESTS

A SIGNIFICANCE TEST CAN BE USED TO DETERMINE IF A DIFFERENCE BETWEEN TWO SETS OF MEASUREMENTS ARE REAL OR DUE TO CHANCE

THE DIFFERENCE BETWEEN TO SETS OF SAMPLES

IS DEEMED <u>SIGNIFICANT</u> IF THE PROBABILITY OF GETTING THE DIFFERENCE IS LESS THAN 5%. (THIS IS AN ARBITARAY PROBABILITY SETTING).

FOR A SAMPLE OF n DRAWN FROM A POPULATION WITH SAMPLE MEAN $\overline{X}$ & POPULATION MEAN μ THE NON-DIMENSIONAL DEVIATION IS GIVEN BY

$$Z_1 = \frac{\overline{X} - \mu}{\sigma_m}$$

SINCE FROM THE DEFINITION OF THE STANDARD ERROR

$$\sigma_m = \sigma / \sqrt{n}$$

σ = POPULATION STANDARD DEVIATION.

WE HAVE

$$Z_1 = \left(\frac{\overline{X} - \mu}{\sigma}\right)\sqrt{n} \, .$$

SINCE σ IS RARELY KNOWN, THE SAMPLE

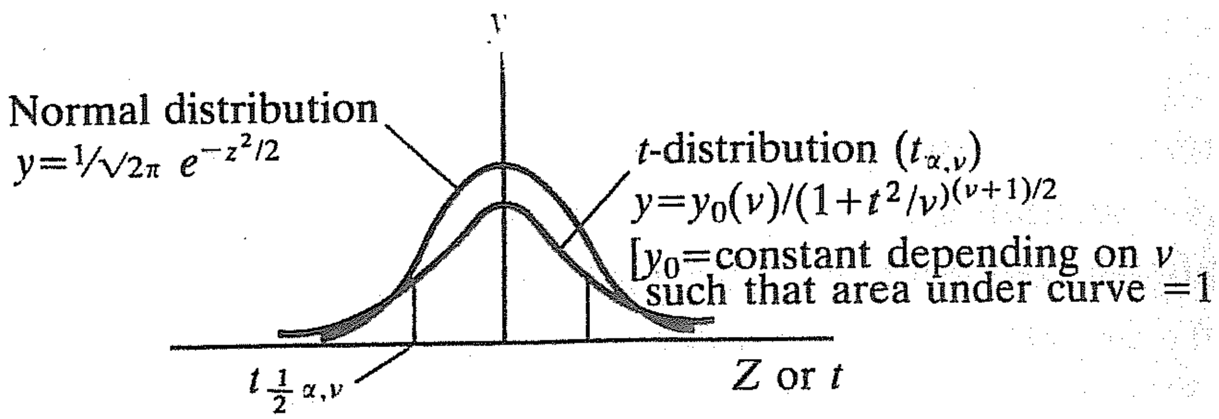
STANDARD DEVIATION S CAN BE USED,

$$t = \frac{\sqrt{n}\,(\bar{X} - \mu)}{S}$$

THE t-DISTRIBUTION FOR LARGE n (>20) IS SIMILAR TO THE GAUSSIAN DISTRIBUTION, FOR SMALL n IT IS DIFFERENT. THE t-DISTRIBUTION TAKES THE FORM

$$Y(\nu) = Y_0(\nu) / \left(1 + t^2/\nu\right)^{(\nu+1)/2}$$

WHERE ν IS THE NUMBER OF THE DEGREES OF FREEDOM ($\nu = n-1$). THE AREA UNDER THE t-DISTRIBUTION REPRESENTS THE PROBABILITY



α

T-Distribution

v	50%	60%	70%	80%	90%	95%	98%	99%	99.5%	99.8%	99.9%	
					0·1	0·05		0·01			0·001	SIGNIFICANCE α
												CONFIDENCE $(1-\alpha)$
1	1	1.376	1.963	3.078	6.314	12.71	31.82	63.66	127.3	318.3	636.6	
2	0.816	1.061	1.386	1.886	2.92	4.303	6.965	9.925	14.09	22.33	31.6	
3	0.765	0.978	1.25	1.638	2.353	3.182	4.541	5.841	7.453	10.21	12.92	
4	0.741	0.941	1.19	1.533	2.132	2.776	3.747	4.604	5.598	7.173	8.61	
5	0.727	0.92	1.156	1.476	2.015	2.571	3.365	4.032	4.773	5.893	6.869	
6	0.718	0.906	1.134	1.44	1.943	2.447	3.143	3.707	4.317	5.208	5.959	
7	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.029	4.785	5.408	
8	0.706	0.889	1.108	1.397	1.86	2.306	2.896	3.355	3.833	4.501	5.041	
9	0.703	0.883	1.1	1.383	1.833	2.262	2.821	3.25	3.69	4.297	4.781	
10	0.7	0.879	1.093	1.372	1.812	2.228	2.764	3.169	3.581	4.144	4.587	
11	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	3.497	4.025	4.437	
12	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.428	3.93	4.318	
13	0.694	0.87	1.079	1.35	1.771	2.16	2.65	3.012	3.372	3.852	4.221	
14	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.326	3.787	4.14	
15	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.286	3.733	4.073	
16	0.69	0.865	1.071	1.337	1.746	2.12	2.583	2.921	3.252	3.686	4.015	
17	0.689	0.863	1.069	1.333	1.74	2.11	2.567	2.898	3.222	3.646	3.965	
18	0.688	0.862	1.067	1.33	1.734	2.101	2.552	2.878	3.197	3.61	3.922	
19	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.174	3.579	3.883	
20	0.687	0.86	1.064	1.325	1.725	2.086	2.528	2.845	3.153	3.552	3.85	
25	0.684	0.856	1.058	1.316	1.708	2.06	2.485	2.787	3.078	3.45	3.725	
30	0.683	0.854	1.055	1.31	1.697	2.042	2.457	2.75	3.03	3.385	3.646	
40	0.681	0.851	1.05	1.303	1.684	2.021	2.423	2.704	2.971	3.307	3.551	
50	0.679	0.849	1.047	1.299	1.676	2.009	2.403	2.678	2.937	3.261	3.496	
60	0.679	0.848	1.045	1.296	1.671	2	2.39	2.66	2.915	3.232	3.46	
80	0.678	0.846	1.043	1.292	1.664	1.99	2.374	2.639	2.887	3.195	3.416	
100	0.677	0.845	1.042	1.29	1.66	1.984	2.364	2.626	2.871	3.174	3.39	
120	0.677	0.845	1.041	1.289	1.658	1.98	2.358	2.617	2.86	3.16	3.373	
Infinity	0.674	0.842	1.036	1.282	1.645	1.96	2.326	2.576	2.807	3.09	3.291	

$\nu = (n-1)$

AS ν INCREASES t-DISTRIBUTION APPROACHES GAUSSIAN.

$t_{0.05,60} \begin{Bmatrix} \alpha = 0.05 \\ \nu = 60 \end{Bmatrix} = 2.00$

$\alpha = 0.05$, OR 95% CONFIDENCE IS MOST USED.

OF A DEVIATION OCCURING. IT IS USUAL TO EXPRESS THE SIGNIFICANCE TEST AS,

$$(\bar{X} - \mu) = \frac{tS}{\sqrt{n}} = \frac{t_{\alpha,\nu} S}{\sqrt{n}}$$

AND WRITING THE PROBABILITY AS $t_{\alpha,\nu}$ WHERE α REPRESENTS AREA UNDER THE t-DISTRIBUTION FROM $(-\infty$ TO $-t_\nu)$ AND FROM $(t_\nu$ to $+\infty)$. α IS KNOWN AS THE SIGNIFICANCE LEVEL.

CONFIDENCE LEVEL

CONFIDENCE IS RELATED TO SIGNIFICANCE IN THAT THE SIGNIFICANCE LEVEL IS RELATED BY $(1-\alpha)$. THUS THE CONFIDENCE LEVEL IS $(1-\alpha)$. THE CONFIDENCE OF A MEASUREMENT SAMPLE IS SUCH THAT THE SAMPLE MEAN DEVIATION $(\bar{X} - \mu)$ WILL NOT BE GREATER THAN

$$\pm \frac{t_{\alpha,\nu} S}{\sqrt{n}}$$

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ERROR & ERROR ANALYSIS

EXAMPLES

A - SIGNIFICANCE TEST

B - CONFIDENCE & UNCERTAINTY

C - SAMPLE NUMBERS.

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AN EXAMPLE OF STATISTICAL DATA ANALYSIS

– SIGNIFICANCE TESTS

GOVERNMENT TESTS OF VEHICLE FUEL CONSUMPTION GIVE 9.7 l/100 km FOR A PARTICULAR MODEL.

TESTS UNDERTAKEN BY A TEST LAB YIELD THE FOLLOWING RESULTS FOR 5-REPEAT TESTS:

9.3, 9.8, 9.1, 9.4 & 9.5

DETERMINE IF THE RESULTS ARE SIGNIFICANTLY DIFFERENT FROM THE PUBLISHED FUEL CONSUMPTION.

HERE WE WISH TO FIND IF

$$|t_A| = \left| \frac{(\bar{X} - \mu)\sqrt{n}}{S} \right| < t_{\alpha,\nu}$$

THAT IS IF $|t_A| < t_{\alpha,\nu}$ THE SAMPLE

IS NOT SIGNIFICANTLY DIFFERENT FROM THE PUBLISHED VALUE. IF $|t_A| > t_{\alpha,\nu}$ THE SAMPLE IS SIGNIFICANTLY DIFFERENT.

FOR A SIGNIFICANCE TEST $\alpha = 0.05$ IS NORMALLY USED (95% CONFIDENCE).

(1) CALCULATE THE SAMPLE MEAN $\bar{X}$

$$\bar{X} = \frac{1}{n} \sum_{k=1}^{n} X_k$$

$n = 5$, $\bar{X} = 9.4$ l/100km.

(2) CALCULATE THE SAMPLE STANDARD DEVIATION

$$S^2 = \left(\frac{1}{n-1}\right) \sum_{k=1}^{n} \varepsilon_k^2 \qquad \varepsilon_k = X_k - \bar{X}$$

$$S^2 = 0.0670$$

$$S = 0.26 \text{ l/100km}$$

(3) FIND t_A

$$t_A = \frac{(\bar{X} - \mu)\sqrt{n}}{S}$$

THE POPULATION MEAN $\mu = 9.7\ \ell/100\text{km}$

THUS $t_A = -2.6$ $\quad |t_A| = 2.6$

(4) FROM THE t- DISTRIBUTION WE FIND,

$$t_{0.05,4} = 2.776 \qquad \begin{cases} \alpha = 0.05 \\ \nu = 4 = (n-1) \end{cases}$$

THUS SINCE $|t_A| = 2.6 < t_{0.05,4} = 2.776$

THE DIFFERENCE BETWEEN THE SAMPLE MEAN OF $9.4\ \ell/100\text{km}$ & POPULATION MEAN OF $9.7\ \ell/100\text{km}$ IS THEREFORE NOT SIGNIFICANT AT THE 0.05 SIGNIFICANCE LEVEL.

- CONFIDENCE & UNCERTAINTY.

USING THE SAME DATA AS ABOVE DETERMINE THE 95% & 99% CONFIDENCE IN THE DATA.

CONFIDENCE LEVEL IS GIVEN BY

$$\bar{X} \pm \frac{t_{\alpha,\nu} S}{\sqrt{n}}$$

THUS FOR

$$t_{0.05,4} = 2.776$$

$$t_{0.01,4} = 4.604$$

WE HAVE FOR $\bar{X} = 9.4\ \ell/100\text{km}$,

$n = 5$, $S = 0.26\ \ell/100\text{km}$.

95% $\bar{X} = 9.4 \pm 0.32\ \ell/100\text{km}$

99% $\bar{X} = 9.4 \pm 0.54\ \ell/100\text{km}$.

THUS WE MAY SAY THAT 1 IN 20

FUEL CONSUMPTION MEASUREMENTS WILL FALL OUTSIDE THE RANGE OF 9.4 ± 0.32 ℓ/100 km (95% CONFIDENCE); AND 1 IN 100 WILL FALL OUT SIDE THE RANGE 9.4 ± 0.54 ℓ/100km. (99% CONFIDENCE).

WE CAN ALSO SAY THAT THE UNCERTAINTY IN THE MEASURED FUEL CONSUMPTION IS ± 0.32 ℓ/100 km AT THE 95% CONFIDENCE LEVEL. (± 0.54 ℓ/100 km AT 99%)

THE UNCERTAINTY (ESTIMATION OF THE ERROR) IS BASED ONLY ON PRECISION ERROR. TOTAL UNCERTAINTY REQUIRES INCLUSION OF THE BIAS ERROR.

SAMPLE SIZE ESTIMATES

GIVEN THE FUEL CONSUMPTION DATA, HOW MANY MEASUREMENTS WOULD BE REQUIRED TO REDUCE THE 95% CONFIDENCE UNCERTAINTY TO ± 0.1 l/100 km?

$$\delta = (t_{0.05,\nu}\, S)/\sqrt{n}$$

FOR $S = 0.26$ l/100 km, $\delta = 0.1$ l/100 km

$$n = 6.67\,(t_{0.05,\nu})^2$$

CLEARLY SINCE $t_{\alpha,\nu} = f(n)$ FOR $\alpha = \text{CONST}$

WE NEED TO ESTIMATE A VALUE & CHECK.

FOR $n = 26$ $\quad t_{0.05,25} = 2.06$

THUS $\quad 6.67\,(2.06)^2 = 28.3 = 28$

FOR $n = 31$ $\quad t_{0.05,30} = 2.042$

THUS $\quad 6.67(2.042)^2 = 27.8 = 28$

THUS AROUND 28 FUEL CONSUMPTION TESTS WOULD NEED TO BE DONE TO REDUCE THE UNCERTAINTY TO ± 0.1 l/100km AT THE 95% CONFIDENCE LEVEL.

FOR LARGE SAMPLES IT IS NORMAL TO USE $t_{0.05,\infty} = 2.0$ (WHICH IS THE GAUSSIAN VALUE FOR 2σ FROM THE MEAN)

THUS

$$n = \left(\frac{2S}{\delta}\right)^2$$

IN THIS EXAMPLE $n = 27$. SUCH A LARGE NUMBER WOULD BE DIFFICULT & EXPENSIVE FOR FUEL CONSUMPTION MEASUREMENTS.

MEC3458 EXPERIMENTAL PROJECTS

ERROR & ERROR ANALYSIS

TOPIC 3: ERROR PROPAGATION*

D. HONNERY 2006

*BASED ON 'SUMMARY OF EXPERIMENTAL UNCERTAINTY ASSESSMENT METHODOLOGY WITH EXAMPLE' BY STERN, MUSTE, BENINATI & EICHINGER. IIHR TECHNICAL REPORT 406 1999.
SEE MUSO FOR THIS DOCUMENT.

ERROR PROPAGATION.

CONSIDER A MEASUREMENT r WHICH IS DERIVED FROM MANY VARIABLES SUCH THAT $r = r(x_1, x_2, x_3, \ldots x_j, \ldots)$.

FOR EXAMPLE MEASUREMENT OF THE DRAG COEFFICIENT

$$C_D = F/(\tfrac{1}{2}\rho v^2 A).$$

TO DETERMINE THIS INDIVIDUAL MEASUREMENT OF

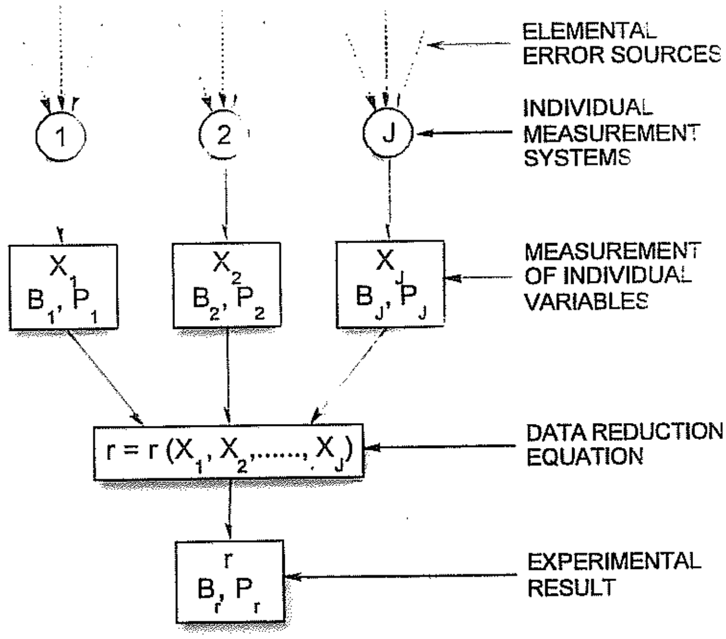
F - FORCE

ρ - DENSITY $<$ TEMPERATURE (T), PRESSURE (p).

v - VELOCITY

A - AREA.

ARE REQUIRED. EACH WILL HAVE A CORRESPONDING UNCERTAINTY CONSISTING OF A BIAS ERROR (B) & A PRECISION ERROR (P). WHEN EACH



PROPAGATION OF EXPERIMENTAL ERROR THROUGH AN EXPERIMENT ERROR - UNCERTAINTY - INCREASES.

B = BIAS ERROR
P = PRECISION ERROR

MEASUREMENT IS COMBINED VIA SOME FUNCTION THE INDIVIDUAL ERRORS TOO MUST COMBINE. ERROR - UNCERTAINTY - WILL INCREASE. THE INCREASE IN UNCERTAINTY IS KNOWN AS ERROR PROPAGATION. THIS SECTION DEALS WITH METHODS FOR ACCOUNTING FOR THIS PROCESS. 11 PROVIDES A METHOD FOR CALCULATION OF THE UNCERTAINTY IN C_D GIVEN UNCERTAINTIES IN F, ρ, v & A ; AND ρ FOR T & p.

PROPAGATION EQUATION

CONSIDER THE SIMPLE CASE OF A FUNCTION OF THE FORM

$$r = r(x, y).$$

WHERE WE LIMIT THE CASE TO TWO MEASUREMENT VARIABLES x & y.

WE CAN RELATE THE k'th MEASUREMENT OF x & y TO THEIR TRUE VALUES AS

$$x_k = x_{TRUE} + \beta_{x,k} + \varepsilon_{x,k}$$

$$y_k = y_{TRUE} + \beta_{y,k} + \varepsilon_{y,k}.$$

WHERE β IS THE BIAS ERROR DEVIATION & ε THE PRECISION ERROR DEVIATION.

NOW CONSIDER THE DEVIATION OF THE k'th VALUE OF THE FUNCTION r FROM ITS TRUE VALUE $(r_k - r_{TRUE})$. THIS CAN BE WRITTEN AS A TAYLOR SERIES EXPANSION,

$$r_k - r_{TRUE} = \frac{\partial r}{\partial x}\left(x_k - x_{TRUE}\right) + \frac{\partial r}{\partial y}\left(y_k - y_{TRUE}\right)$$

HERE ONLY FIRST ORDER TERMS ARE CONSIDERED.

DEFINING $\delta_{r_k} = r_k - r_{TRUE}$

& INTRODUCING SENSITIVITY COEFFICIENTS

$$\theta_x = \frac{\partial r}{\partial x} \qquad \theta_y = \frac{\partial r}{\partial y}$$

WE HAVE

$$\delta_{r_k} = \theta_x\left(\beta_{x_k} + \varepsilon_{x_k}\right) + \theta_y\left(\beta_{y_k} + \varepsilon_{y,k}\right)$$

WE NOW WISH TO DETERMINE THE DISTRIBUTION OF δ_r FOR $1, 2, 3, k, \ldots N$ MEASUREMENTS OF r. NAMELY WE WANT THE STANDARD DEVIATION OF δ_r. WRITTEN AS THE SQUARE OF THE DEVIATION (VARIANCE) THIS IS,

$$\sigma_{\delta_r}^2 = \frac{1}{N} \sum_{k=1}^{N} \left(\delta_{r_k}\right)^2$$

(NOTE $N \to \infty$. HENCE IS A LIMIT.)

EACH ERROR TERM $(\beta_{x_k}, \varepsilon_{x,k}, \beta_{y_k}, \varepsilon_{y_k})$ WILL HAVE A CORRESPONDING STANDARD DEVIATION SUCH THAT

$$\sigma_{\delta r}^2 = \Theta_x^2 \sigma_{\beta x}^2 + \Theta_y^2 \sigma_{\beta_y}^2 + 2\Theta_x \Theta_y \sigma_{\beta_x \beta_y} + \Theta_x^2 \sigma_{\varepsilon_x}^2 + \ldots$$

$$+ \Theta_y^2 \sigma_{\varepsilon_y} + 2\Theta_x \Theta_y \sigma_{\varepsilon_x \varepsilon_y}.$$

HERE σ_β^2 & σ_ε^2 ARE THE VARIANCE OF THE BIAS & PRECISION ERRORS & $\sigma_{\beta_x\beta_y}$ & $\sigma_{\varepsilon_x\varepsilon_y}$ THE COVARIANCE OF THE ERROR DISTRIBUTIONS.

HOWEVER WE RARELY KNOW THESE POPULATION BASED STATISTICS SO WE INTRODUCE ESTIMATES VIA UNCERTAINTIES.

THUS WE INTRODUCE

u_c^2 FOR $\sigma_{\delta_r}^2$

b_x^2, b_y^2 & b_{xy} FOR $\sigma_{\beta x}^2, \sigma_{\beta y}^2$ & $\sigma_{\beta_x\beta_y}$

S_x^2, S_y^2 & S_{xy} FOR $\sigma_{\varepsilon_x}^2, \sigma_{\varepsilon_y}^2$ & $\sigma_{\varepsilon_x\varepsilon_y}$

THUS THE ESTIMATE OF THE TOTAL ERROR IS

$$u_c^2 = \theta_x^2 b_x^2 + \theta_y^2 b_y^2 + 2\theta_x\theta_y b_{xy} + \theta_x^2 S_x^2 + \theta_y^2 S_y^2 + 2\theta_x\theta_y S_{xy}$$

AS WITH ALL UNCERTAINTIES THE UNCERTAINTY MUST BE EXPRESSED AT A PARTICULAR CONFIDENCE LEVEL. 95% IS USUAL (t-DISTRIBUTION $(1-\alpha) = 95\%$, $t \rightarrow 2$). UNCERTAINTY THEN BECOMES,

$$U_r = k u_c$$

WHERE K IS A 'COVERAGE FACTOR' RESULTING FROM THE CONFIDENCE LEVEL USED. FOR J MEASUREMENTS WE CAN WRITE THE TOTAL UNCERTAINTY AS,

$$U_r^2 = \underbrace{\sum_{i=1}^{J} \theta_i^2 B_i^2 + 2\sum_{i=1}^{J-1} \sum_{k=i+1}^{J} \theta_i \theta_k B_{ik}}_{B_r^2} + \underbrace{\sum_{i=1}^{J} \theta_i^2 P_i^2 + 2\sum_{i=1}^{J-1} \sum_{k=i+1}^{J} \theta_i \theta_k P_{ik}}_{P_r^2}$$

WHERE

$$\left.\begin{aligned} B_i &= t b_i \\ B_{ik} &= t^2 b_{ik} \\ P_i &= t S_i \\ P_{ik} &= t^2 S_{ik} \end{aligned}\right\}$$

($t=2$ FOR 95% CONFID.).

& $B_{ik} = \sum_{l=1}^{L} (B_i)_l (B_k)_l$

B_i & P_i ARE THE BIAS & PRECISION LIMITS IN x_i & B_{ik}, P_{ik} ARE THE CORRELATED BIAS & PRECISION LIMITS IN x_i & x_k.

WE MAY NOW WRITE A GENERAL EQUATION AS

$$U_r^2 = B_r^2 + P_r^2 \qquad \text{(SEE ABOVE).}$$

DETERMINING THE VALUES FOR B_r & P_r WILL DEPEND ON MANY FACTORS.

<u>BIAS ERRORS</u> DEPEND ON, FOR EXAMPLE, THE RESOLUTION OF THE MEASUREMENT DEVICE & ARE THEREFORE INDEPENDENT OF THE NUMBER OF SAMPLES (REPEAT TESTS). <u>PRECISION ERRORS</u> <u>WILL</u> DEPEND ON HOW MANY TEST WERE DONE.

PRECISION LIMITS FOR A SINGLE TEST

A TEST MAY BE SINGLE OR MULTIPLE.

A SINGLE TEST IS A TEST DONE AT ONE PARTICULAR INSTANT. A REPEAT IS ANOTHER SINGLE, BUT INDEPENDENT, TEST DONE AT THE SAME CONDITIONS AS ANOTHER TEST.

A SINGLE TEST COULD BE A MEASUREMENT DONE OVER A TIME PEROID RESULTING IN AN AVERAGE FOR THAT MEASUREMENT. EG YOU MEASURE TEMPERATURE FOR 10 MINUTES EVERY SECOND & DETERMINE THE AVERAGE (& STANDARD DEVIATION).

A REPEAT MEASUREMENT WOULD BE AN AVERAGE

FROM A SECOND 10 MINUTE PERIOD.

IGNORING THE CORRELATION (COVARIANT TERMS) TERMS WE CAN WRITE THE PRECISION ERROR P_r FOR A SINGLE MEASUREMENT AS

$$P_r = t S_r$$

TYPICALLY $t = 2$ THUS FOR 95% CONFIDENCE

$$P_r = 2 S_r$$

WHERE S_r IS THE STANDARD DEVIATION OF THE MEASUREMENT.

FOR THE CASE OF MANY MEASUREMENT VARIABLES (EG FORCE, AREA, VELOCITY ETC IN DETERMINING C_D),

$$P_r^2 = \sum_{i=1}^{J} (\theta_i P_i)^2 \quad , \quad P_i = t S_i$$

IN THE CASE WHERE ONLY A SINGLE MEASUREMENT IS RECORDED (IE A TEMPERATURE MEASUREMENT AT A PARTICULAR TIME) THE STANDARD DEVIATION MUST EITHER BE KNOWN OR DETERMINED. EG. YOU COULD RECORD A NUMBER OF MEASUREMENTS OF TEMPERATURE DURING THE SINGLE TEST, OR THE DATA COULD BE AVAILABLE.

PRECISION LIMITS FOR MULTIPLE TESTS.

FOR MULTIPLE TESTS THE SAMPLE STANDARD DEVIATION OF THE TESTS IS REQUIRED. SAY WE HAD M- MEASUREMENTS (WE MEASURED TEMPERATURE EACH DAY FOR M-DAYS).

FOR THIS WE MAY THEN HAVE

AN AVERAGE $\bar{r}$ FOR THE M- MEASUREMENTS.

IF EACH MEASUREMENT HAD A STANDARD DEVIATION OF $S_{\bar{r}}$

$$\text{THUS} \quad P_r = t S_{\bar{r}}$$

FOR THE MULTIPLE SAMPLES

$$P_{\bar{r}} = \frac{t S_{\bar{r}}}{\sqrt{M}}$$

NOTE: t DEPENDS ON M (IN FACT ν SINCE $\nu = M-1$).

$$P_{\bar{r}} \approx \frac{2 S_{\bar{r}}}{\sqrt{M}}$$

FOR $M \geq 10$
$t \to 2$

THUS PRECISION ERROR FALLS AS MEASUREMENT NUMBER INCREASES.

THUS WE HAVE FOR THE TOTAL UNCERTAINTY

$$U_c^2 = B_r^2 + P_{\bar{r}}^2 = B_r^2 + \left(\frac{t S_{\bar{r}}}{\sqrt{M}}\right)^2$$

FOR THE CASE OF MANY MEASUREMENT VARIABLES

$$P_{\bar{r}}^2 = \sum_{i=1}^{j} \left(\theta_i \, P_{\bar{i}}\right)^2$$

WHERE $P_{\bar{i}} = \dfrac{t S_{\bar{i}}}{\sqrt{M}}$ IS FOR ONE VARIABLE.

THE METHOD OF APPLICATION IS BEST LEFT TO AN EXAMPLE.

MEC3458 EXPERIMENTAL PROJECTS

ERROR & ERROR ANALYSIS

EXAMPLES E – MULTIPLE TEST TWO VARIABLE

F – SINGLE TEST TWO VARIABLE

G. – MODIFIED PRECISION ESTIMATE

– COMMENTS.

D. HONNERY 2008.

ERROR PROPAGATION - TWO VARIABLE TEST

AN EXPERIMENT TO DETERMINE THE FORCE ACTING ON A MASS UNDERGOING AN ACCELERATION RESULTS IN THE FOLLOWING.

(1) VARIABLES

MASS - m. 2.000 ± 0.001 kg.

THIS A CALIBRATED MASS WITH QUOTED ERROR AT THE 95% CONFIDENCE LEVEL.

ACCELERATION - a - MEASURED BY AN ACCELEROMETER WITH RESOLUTION 0.01 m/s^2.

(2) RELATIONSHIP

THE RELATIONSHIP FUNCTION IS

$$F = m.a \quad (N).$$

$\{ r = r(x, y)$

(3) RESULTS.

10 TEST WERE UNDERTAKEN RESULTING IN THE FOLLOWING ACCELERATIONS.

9.71, 9.69, 9.75, 9.77, 9.80, 9.75, 9.69, 9.73
9.72, 9.75.

THE MEAN & STANDARD DEVIATION ARE:

$\bar{a} = 9.74 \ m/s^2$ $\qquad \bar{F} = 19.48 \ N$

$S_{\bar{a}} = 0.033 \ m/s^2$ $\qquad S_{\bar{F}} = 0.066 \ N$

(4) UNCERTAINTY

TOTAL UNCERTAINTY IS GIVEN BY

$$U_{\bar{F}}^2 = B_F^2 + \left(t\, S_{\bar{F}} / \sqrt{M}\right)^2$$

BIAS LIMITS ARE GIVEN BY;

$$B_F^2 = \theta_m^2 B_m^2 + \theta_a^2 B_a^2$$

THERE WILL BE NO CORRELATION TERMS (B_{ik}) SINCE M & a ARE MEASURED WITH DIFFERENT INSTRUMENTS (IF THERE WERE TWO ACCELERATION COMPONENTS MEASURED WITH THE SAME ACCELEROMETER THERE WOULD BE).

SENSITIVITY TERMS θ_m & θ_a ARE

$$\theta_{\bar{m}} = \frac{\partial F}{\partial M} = \bar{a} \qquad \theta_{\bar{a}} = \frac{\partial F}{\partial a} = \bar{m}.$$

BIAS TERMS B_m & B_a ARE

B_m IS GIVEN AS 0.001 kg.

B_a IS TAKEN TO BE 1/2 THE RESOLUTION OF THE INSTRUMENT.

$B_a = 0.005\ m/s^2$.

THUS USING THE <u>AVERAGE VALUES</u> FOR a & THE CONST M VALUE

$\bar{a} = 9.74\ m/s^2 \quad \bar{M} = M = 2.000\,kg.$

$$B_F^2 = (9.74 * 0.001)^2 + (2 * 0.005)^2$$

$$B_F^2 = 1.95 \times 10^{-4}\ N^2$$

PRECISION LIMITS ARE, FOR 10 TESTS $t \to 2$.

$$P_{\bar{F}}^2 = \left(2 S_{\bar{F}} / \sqrt{M}\right)^2$$

FOR LESS THAN 10 REPEATS, GET t FROM t-TABLE FOR ν.

$M = 10 \qquad S_{\bar{F}} = 0.066\,N \qquad P_{\bar{F}}^2 = 1.74 \times 10^{-3}\ N^2.$

TOTAL UNCERTAINTY IS.

$$U_c = \left(1.95 \times 10^{-4} + 1.74 \times 10^{-3}\right)^{1/2}$$

$$= 0.044\,N.$$

THUS THE AVERAGE FORCE IS

$$\bar{F} = \underline{19.48 \pm 0.044\,N.}$$ @ 95% CONFIDENCE LEVEL.

$\{\bar{F} = 19.48\,N \pm 0.23\%\}$

NOTE THAT THE BIAS ERROR ESTIMATE IS THE SMALLER OF THE TWO.

- SINGLE TEST

CONSIDER NOW THIS EXPERIMENT AS A SINGLE TEST. TAKE THE FIRST MEASUREMENT.

TOTAL UNCERTAINTY IS GIVEN BY

$$U_F^2 = B_F^2 + P_F^2$$

BIAS LIMITS ARE CALCULATED USING THE VALUES OF THE FIRST MEASUREMENT

$$B_F^2 = \theta_m^2 B_m^2 + \theta_a^2 B_a^2$$

$$\theta_m = \frac{\partial F}{\partial m} = a_1 \qquad \theta_a = \frac{\partial F}{\partial a} = m_1$$

NOTE: AVERAGE $\bar{m}$ & $\bar{a}$ ARE NOT USED; m_1, a_1 ARE.

$$a_1 = 9.71\ m/s^2 \qquad m_1 = 2.0\ kg. \quad (m_1 = m)$$

$$B_m = 0.001\ kg \qquad B_a = 0.005\ m/s^2$$

$$B_F^2 = 1.94 \times 10^{-4}\ N^2$$

PRECISION LIMITS ARE CALCULATED VIA A SINGLE TEST BUT USING ESTIMATED S_F.

$$P_F^2 = (2S_F)^2$$

NOTE HERE WE USE $t=2$.

S_F IS SET EQUAL TO $S_{\bar{F}}$ AS AN ESTIMATE WHICH MUST BE KNOWN.

THUS FOR $S_F = S_{\bar{F}} = 0.066$ N

$$P_F^2 = 0.0174 \text{ N}^2$$

TOTAL UNCERTAINTY IS

$$U_F = \left(1.94 \times 10^{-4} + 0.0174\right)^{1/2}$$

$$= 0.133 \text{ N}$$

THUS FOR FIRST MEASUREMENT UNCERTAINTY IS

$$F_1 = 19.42 \pm 0.133 \text{ N}$$

$$\{ F_1 = 19.42 \text{ N} \pm 0.68\% \}$$

MEASUREMENT UNCERTAINTY INCREASES IN A SINGLE TEST AS EXPECTED.

- <u>MULTIPLE TEST WITH SEPARATE PRECISION LIMIT CALCULATION</u>

PRECISION LIMIT CAN BE ESTIMATED BY CONSIDERATION OF THE INDIVIDUAL VARIABLES.

$$P_{\bar{F}}^2 = \theta_m^2 P_{\bar{m}}^2 + \theta_a^2 P_{\bar{a}}^2$$

ONE AGAIN THERE ARE NO CORRELATION TERMS.

FOR $\theta_m = \frac{\partial F}{\partial m} = a$ $\qquad \theta_a = \frac{\partial F}{\partial a} = m$

FOR MASS m $P_{\bar{m}} = 0$ SINCE THIS A CONSTANT WITH KNOWN BIAS LIMITS.

FOR ACCELERATION $P_{\bar{a}}^2 = (2 S_{\bar{a}} / \sqrt{M})^2$

NOTE THAT $S_{\bar{a}}$ IS USED (NOT $S_{\bar{F}}$).

$$S_{\bar{a}} = 0.033 \, m/s^2$$

THUS $$P_{\bar{F}}^2 = 0 + 2^2(2 \times 0.033/\sqrt{10})^2$$

$$P_{\bar{F}}^2 = 1.74 \times 10^{-3} \; N^2$$

BIAS LIMITS WILL BE THE SAME

$$B_{\bar{F}}^2 = 1.94 \times 10^{-4} \; N^2$$

TOTAL UNCERTAINTY IS

$$U_C = (1.94 \times 10^{-4} + 1.74 \times 10^{-3})^{1/2}$$

$$U_C = 0.044 \; N$$

THUS $$\bar{F} = 19.48 \pm 0.044 \, N.$$

WHICH IS THE SAME AS TREATING THE ERROR AS ERROR IN $\bar{F}$ RATHER THAN IN $\bar{m}$ & $\bar{a}$. BOTH METHODS SHOULD YIELD SIMILAR RESULTS.

COMMENTS

(1) BIAS LIMITS ARE GENERALLY SMALLER THAN PRECISION LIMITS — OR SHOULD BE FOR ACCURATE MEASUREMENTS (RELATIVE ERROR).

(2) IF ONLY A SINGLE TEST IS DONE DATA ON THE STANDARD DEVIATION OF THE VARIABLES IS NEEDED. IF YOU DON'T HAVE IT YOU SHOULD REPEAT YOUR MEASUREMENTS (AT LEAST 3 TIMES - REMEMBER THOUGH THAT $t=2$ FOR $M \geq 10$ SO FOR $M=3$ $t=2$ IS A GROSS APPROXIMATION).

(3) BIAS LIMITS CAN BE ESTIMATED AS EITHER ½ THE SMALLEST RESOLUTION (FOR AN 'ANALOGUE' INSTRUMENT) OR AS THE LEAST SIGNIFICANT FIGURE (FOR A 'DIGITAL' INSTRUMENT).

(4) CORRELATION TERMS IN BIAS LIMITS EXIST FOR VARIABLES WITH COMMON MEASUREMENT INSTRUMENTS

SEE THE PAPER BY STERN, MUSTE, BENINATI & EICHINGER FOR AN EXCELLENT BUT MORE COMPLEX EXAMPLE.

MEC3458 EXPERIMENTAL PROJECTS

ERROR & ERROR ANALYSIS

TOPIC 4: DATA PRESENTATION

– SOME OBSERVATIONS

– SIGNIFICANT FIG's.

– GRAPHING DATA

– TRENDS, UNCERTAINTY & DIFFERENCE.

D. HONNERY 2008.

SIGNIFICANT FIGURES (SIG FIGS)

SIGNIFICANT FIGURES RELATES TO THE UNCERTAINTY (PRECISION) OF A MEASUREMENT.

THE NUMBER 1234.5 HAS 5 SIGNIFICANT FIGURES, AS DOES 0.12345, 12345 & 1.2345×10^4. 1.23450 COULD HAVE 5 OR 6, THE TRAILING ZERO WILL ONLY BE SIGNIFICANT IF IT CAN BE SHOWN TO REPRESENT THE PRECISION OF THE NUMBER (MEASUREMENT).

FOR EXAMPLE IF A QUANTITY CAN ONLY BE MEASURED VIA A TECHNIQUE ONLY ABLE TO, SAY, $\pm 5\%$ UNCERTAINTY IT MAKES LITTLE SENSE TO QUOTE THE NUMBER AS $1.2345 \pm 5\%$ SINCE $\pm 5\%$ IS ± 0.06. BY ROUNDING OFF THE UNNECESSARY DIGITS THE MEASUREMENT IS BETTER REPRESENTED BY $1.23 \pm 5\%$.

THERE ARE MANY RULES REGARDING USE OF SIGNIFICANT FIG'S. (CHECK FOR EXAMPLE WIKIPEDIA) HOWEVER THERE IS REALLY ONLY ONE REQUIRED.

'THE NUMBER OF DIGITS USED TO REPRESENT A MEASURED QUANTITY SHOULD REFLECT THE ACCURACY OF THE INSTRUMENT USED & PRECISION OF THE METHOD'

THUS A THERMOMETER WITH SCALE IN 1/10th °C COULD IF READING ROOM TEMP. YIELD A MEASUREMENT OF 20·1°C. IT COULD NOT YIELD A MEASUREMENT OF 20·1453°C. EVEN WITH MANY REPEATS YOU COULD ONLY EVER USE A MAX OF 4 SIG FIGS. & IN ALL LIKELYHOOD THE STATEMENT OF THE MEASUREMENT UNCERTAINTY WOULD PROVIDE MORE MEANING THAN PROVISION OF THE CORRECT NUMBER OF

SIGNIFICANT FIG's.

WHEN CALCULATING QUANTITIES FROM MEASURED VALUES ROUND OFF THE CALCULATED QUANTITY TO A NUMBER OF SIGNIFICANT FIGURES THAT BEST REPRESENTS THOSE OF THE MEASURED QUANTITIES.

GRAPHING DATA – THE DOES & DON'TS.

1. GRAPHS ARE USED WHEN YOU WISH TO SHOW A RELATIONSHIP BETWEEN QUANTITIES. IF A TABLE DOES THIS BETTER – USE A TABLE!
2. GRAPHS MUST BE ANNOTATED, THE UNITS MUST BE GIVEN (UNLESS DIMENSIONLESS).
3. IF PLOTTING MEASURED QUANTITIES, USE SYMBOLS. ONLY DRAW A LINE CONNECTING THE DATA (POINT TO POINT, NOT SMOOTHED) IF YOU WISH TO MAKE THE TREND CLEAR.

WHEN YOU DRAW A LINE CONNECTING MEASURED QUANTITIES ON A GRAPH YOU ARE INDICATING THE POINTS ARE RELATED, OR THEY EXHIBIT A TREND. EITHER WAY THERE MUST BE SOME JUSTIFICATION, PHYSICAL OR OTHERWISE, FOR THE RELATIONSHIP OR TREND - <u>SEE TRENDLINES BELOW</u>

(4) WHEN GRAPHING FUNCTIONS (EG YOU MIGHT BE COMPARING A THEORETICAL RESULT TO MEASUREMENTS) USE A LINE FOR THE FUNCTION & SYMBOLS FOR MEASUREMENTS. <u>ONLY</u> USE A LINE FOR MEASUREMENTS (IE NOT SYMBOLS) WHEN THE DATA IS LARGE & CLOSELY SPACED.

(5) DO NOT FIT SMOOTHED LINES TO EXPERIMENTAL DATA (THIS A MUCH MISUSED PLOTTING OPTION

IN EXCEL - SWITCH IT OFF OR SELECT POINT TO POINT GRAPH OPTION).

(6) IF FITTING A "TRENDLINE" TO DATA YOU ARE SAYING THE RELATIONSHIP BETWEEN THE DATA TAKES THIS FUNCTIONAL FORM (EG LINEAR, QUADRATIC ETC). IN ANY CASE THE FUNCTION USED MUST MAKE PHYSICAL SENSE. ONLY FIT A "TREND LINE" WHEN YOU NEED A REPRESENTIVE FUNCTION FOR THE RELATIONSHIP BETWEEN THE DATA. IF NOT DON'T FIT ONE!

(7) YOU WILL OFTEN NEED TO INDICATE THE UNCERTAINTY OF THE MEASUREMENT ON A GRAPH, MOST PLOTTING PROGRAMS HAVE THIS OPTION. DO THIS WITH SOME CARE. NOT EVERY DATA POINT NEEDS THIS LEVEL OF DETAIL, TO MUCH INFORMATION

IN THE PLOT MAY RENDER IT IMPOSSIBLE TO READ. REMEMBER ALSO THAT BOTH THE INDEPENDENT & DEPENDENT (x & y) VARIABLES COULD HAVE UNCERTAINTY ASSOCIATED WITH THEM ( ).

TRENDS, UNCERTAINTY & DIFFERENCE

SUPPOSE YOU MEASURE TWO QUANTITIES & THE DIFFERENCE BETWEEN THEM IS SMALLER THAN THE UNCERTAINTY RANGE OF THE MEASUREMENTS.

(A) 10.5 ± 0.8 (B) 11.1 ± 0.7

ARE THESE DIFFERENT, OR ARE THEY SIMILAR?

WE HAVE ALREADY NOTED THAT A SIGNIFICANCE TEST CAN BE USED TO DETERMINE WHETHER A SAMPLE MEAN IS SIGNIFICANTLY DIFFERENT FROM A POPULATION MEAN, AND THIS COULD BE DONE HERE. WITH OUT SUCH A TEST (OR OTHER APPROPIATE STATISTICAL TEST) ALL THAT CAN BE SAID IS THAT (MORE) REPEAT MEASUREMENTS WILL BE REQUIRED

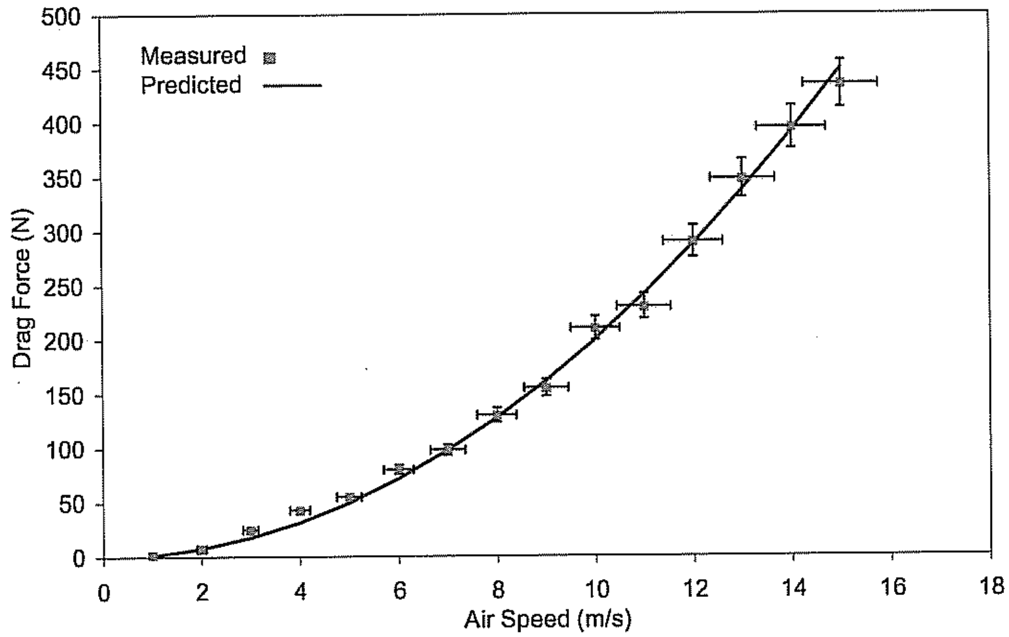
TO DETERMINE WHETHER THE DIFFERENCE IS REAL OR JUST PART OF MEASUREMENT VARIABILITY.

WHAT IF NOW A LARGE SERIES OF MEASUREMENTS EXHIBITED A TREND THAT DIFFERED FROM A SECOND SET, BUT THE DIFFERENCE IN THE TREND, WHILE CONSISTENT, FELL WITH IN THE UNCERTAINTY RANGE OF THE MEASUREMENT?

ONCE AGAIN, ALL THAT CAN BE SAID WITHOUT APPLYING SOME STATISTICAL TEST, IS THAT A CONSISTENT TREND WAS OBSERVERED IN DATA SET (A) THAT DIFFERED (OR WAS SIMILAR TO) DATA SET (B), BUT WITH OUT FURTHER MEASUREMENT, THE SIGNIFICANCE OF THE DIFFERENCE (ETC) CANNOT BE ASSESSED. (COMMENTED ON). JUST BECAUSE THE UNCERTAINTY IS LARGE DOES NOT MEAN THE DIFFERENCE (IN VALUE OR TREND) IS NOT <u>REAL</u> - YOU JUST CAN'T BE <u>SURE</u>.

Sample Graphs

Below is a well constructed graph. Measured data is shown as symbols and error bars (+/- 5%) for both air speed and drag force are given. As you can see if you have many data points error bars could become difficult to read. Predicted data is shown as a sold curve. A legend is given and units provided for each quantity.



This graph shows two sets of experimental data each with different point to point connector lines and symbols. Once again the graph is fully annotated.